7MMA2E1

Discrete mathematics:-

Unit - I Algebraic system: Binary operation - Algebraic systems - semigroup and monoids - Homomorphism of Semigroup and monoids - properties of Homomorphism - subsemi groups and Submonoids.

Unit - II
Mathematical induction - Technique of proof - Mathematical induction - Recurrence Relations and Generating function - Recurrence - an introduction - polynomial and their Evaluations Recurrence Relations - SOl of Finite order Homogeneous (Linear) Relations

Unit - III
Solution of Non-homogeneous relation. Generation function - Some common Recurrence Relations - Primitive Recursive function - Recursive and partial Recursive function.

Unit - IV
Lattices - Lattices - Some properties of Lattices - New Lattices - Modular and Distributive Lattices.

Unit V

Boolean Algebra – Boolean Algebras – Boolean polynomials – Karnaugh map – Switching circuits.

Text Book:-

1. Dr. M.K. Venkatraman, Dr. N. Sridharan & Dr. N chandra sekaran, The National publishing Company, Chennai

chapter IV: chapter V – Sections 1 to 9

chapter VII – Sections 7.1 to 7.6 chapter X

9360764248

Prof: Amsajothi

Unit-I

Algebraic systems:-

Def

An algebraic system is a mathematical system consisting of a set and one or more n-ary operations on the set. It is denoted by $(S, f_1, f_2 \ldots)$ where S is a non-empty set and $f_1, f_2 \ldots$ are operations on S. Since the operations define a structure on the statements of S an algebraic system is called an algebraic structure.

Ex:-

1. $(N, +, \times)$ is an algebraic system. Where N is the set of natural numbers and the operations + and × have the usual meanings.

Def (Semi groups)

A non-empty set S together with an associative binary operation

$*$ on S. 2

Thus $(S,*)$ is a semi group, if for any $a,b,c \in S$, $(a*b)*c = a*(b*c)$

The semi group $(S,*)$ is said to be commutative, if $*$ is a commutative operations.

Ex:-

1. let N be the set of positive integers. the $(N,+)$ and $(N,\times)$ are semi group since addition and multiplication on N are associative. Thus semi group are also commutative

Def: (Monaid)

A semi group $(M,*)$ with an identity element is called a monaid.

Thus $(M,*)$ is monoid if

(1) for any $a,b,c \in S$ $(a*b)*c = a*(b*c)$

(2) $\exists$ an element $e \in M \ni$ for any $x \in M$, $x*e = e*x = x$.

Thus the monoid has a unique or a special elements called its identity

Ex:

(i) $(Z,+)$ is a Commutative Semi-group having the number 0 is the identity elements. Hence $(Z,+)$ is a monoid.

Def (Invertible)

Let $(M,*,e)$ be a monoid and $a \in M$. Then if there exist an element $b \in M$ $\ni$: $a*b = b*a = e$. Then b is called an inverse of a. In this case a is said to be invertible.

Theorem:

Let $(M,*,e)$ be a monoid and $a \in M$. If a is invertible then its inverse is unique.

Pf: Let a be an invertible element in $(M,*,e)$. Let b' and b'' be

A be element of M such that

$a * b' = b' * a = e \longrightarrow (1)$

$a * b'' = b'' * a = e \longrightarrow (2)$

Now, $b' = b' * e$

$= b' * (a * b'')$ from (2)

$= (b' * a) * b''$ by (associative law)

$= e * b''$ by (1)

$= b''$

Then $b' = b''$ and hence the inverse of a

it is exist is unique.

Def (Cyclic monoid)

A monoid $(M, *, e)$ is said to be cyclic. If there exist an element $a \in M$ such that every element of M can be written as same power of a. In such a case (i.e) a^n for some $n \in N$. In such a case the cyclic monoid is said to be generated by the element a. The element is called a generator of the cyclic monoid.

Q5 Let N be the set of +ve integer and $*$ the operation of least common multiple on N. Find whether $(N, *)$ is a commutative semigroup. Is it a monoid? Specify the identity element which element in N have inverse and what are they?

Sol: Let $a, b, c \in N$, let A be the set of all prime numbers, which devide atleast one of the numbers a, b, c.

No $A = \{ p : p$ is a prime num. and p divided atleast one of $a, b, c \}$

Then A is a finite set, let,

Let $A = \{ p_1, p_2 \ldots p_n \}$ we can write

$a = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \ldots p_m^{\alpha_m}$, $b = p_1^{\beta_1} \cdot p_2^{\beta_2} \ldots p_m^{\beta_m}$

$c = p_1^{\gamma_1} p_2^{\gamma_2} \ldots p_m^{\gamma_m}$ for some non-negative integer $\alpha_1, \alpha_2 \ldots \alpha_m, \beta_1, \beta_2 \ldots \beta_m$

$\gamma_1, \gamma_2 \ldots \gamma_m$

Then $a * b = L.C.M\,(a, b)$

$= p_1^{e_1} p_2^{e_2} \ldots p_m^{e_m}$

where $\gamma_i = \max\{\alpha_i, \beta_i\}$ $\forall$ $i = 1, 2, \ldots m$.

As $\max\{\alpha_i, \beta_i\} = \max\{\beta_i, \alpha_i\}$ $\forall i$

we have $a * b = b * a$

As $\max\{\max\{\alpha_i, \beta_i\}, \gamma_i\} = \max\{\alpha_i, \beta_i, \gamma_i\}$

$= \max\{\alpha_i, \max\{\beta_i, \gamma_i\}\}$

we have $(a*b)*c = a*(b*c)$

Thus $*$ is associative and $(N, *)$ is a conversetative semigroup.

Since, $a*1 = 1*a = a$ for all $a \in N$

1 is the identity element for $*$ in A

Thus $(N, *)$ is a monoid.

Let $a, b \in N$ such that $a * b = 1$

Then as 1 is the L.C.M of a & b

we've $a \leq 1$, $b \leq 1$. But as $a, b \in N$

we've $a = 1$ & $b = 1$

Thus has inverse with respect to

A.

Q) Show that for a finite monoid $(M, *)$ no two rows or columns of the cayley tables are identical.

Sol: As M is a monoid it has an identity elem, denote it by a_1 & the remaining elem of M as $a_2 \ldots a_n$ (n denotes the number of elements of m)

Then the first row of the composition table is $a_1, a_2 \ldots a_n$ which is also the 1^{st} column obviously no two rows & columns are identical since their 1^{st} elements are different.

Def **Homomorphism:**

Let $(S, *)$ & $(T, \circ)$ be an two semigroups. A mapping $g: S \to T$ such that for any two elements.

$a, b, c \in S$, $g(a * b) = g(a) \Delta g(b)$ is called a semigroup homomorphism.

Suppose g is also one to one and onto. Then g is called an <u>isomorphism</u>.

<u>Ex:</u>

$g : \Gamma \to Z$ defined by two eq:

$g(f^i \circ f^j) = g(f^i) + g(f^j)$ is 1-1 & onto and a homomorphism and g is an isomorphism.

<u>Ex: 1</u>

Show that there exist a homomorph from the algebraic system $(N, +)$ to the system $(Z_4, +_4)$ where

(i) N is the set of natural num!

& (ii) Z_4 is the set of integers modulo 4. if an homomorphism ?.

<u>Sol:</u> let us define $g : N \to Z_4$ by

$g(a) = \{ a \pmod 4 \}$ for all $a \in N$.

for $a, b \in N$, let $g(a) = [i]$ and $b \notin [i]$

Then $g(a+b) = [(i+j)(\text{mod } 4)]$

$= [i] +_4 [j]$

$= g(a) +_4 g(b)$

$\therefore g$ is a homomorphism

As the mapping defined by g is not 1-1

It's not isomorphism.

Ex: 2 S.T. the mapping g from the algebraic system $(S, +)$ to the system $(T, \times)$ defined by $g(a) = 3^a$, where (i) S is the set of all rational numbers under addition operation + & $\times$ (ii) T is the set of non-zero real numbers under multiplication operation $\times$, is a homomorphism but not isomorphism.

Sol: Now $g(a) = 3^a$ for any $a \in S$

(Not $g(a) > 0$ for all a in S)

Suppose $a, b \in S$.

Then $g(a+b) = 2^{a+b}$

$= 2^a \cdot 2^b$

$= g(a) \cdot g(b)$

Hence the mapping by g is a homomor

Suppose $g(a) = g(b)$ Then $2^a = 2^b$ which implies $a = b$

Hence g is 1-1. g is not onto since Range (g) contains no negative numbers.

Hence, g is not an isomorphism.

Ex: 3

Let R^+ be the set of positive real numbers. Show that the function $g: R^+ \to R$ defined by $g(x) = \log_e x$ is an isomorphism of the semigroup $(R, \times)$ to the semigroup $(R, +)$ where $\times$ and $+$ are usual multiplication & addition respectively.

Soln:

Let $x, y \in R$.

It's given that $g(x) = \log_e x$

$g(x \cdot y) = \log(x \cdot y) = \log(x) + \log(y)$

$= g(x) + g(y)$

Hence the function is a homomorphism

Part g is onto take $y \in R$ then

$y = \log_e y = g(e^y)$

Hence g is onto

Suppose $\log x = \log y$

Then $e^{\log x} = e^{\log y}$ (i.e.) $x = y$

Hence the mapping g is 1-1

So the function g is an homomorphism

Ex 4 Let T be the set of all even integers. Show that the semigroups $(Z, +)$ & $(T, +)$ are isomorphic.

Soln Step: 1

We define the function

$g : Z \to T$ given by $g(a) = 2a$

where $a \in Z$

Step: 2

Suppose $g(a_1) = g(a_2)$ where $a_1, a_2 \in$

Then $2a_1 = 2a_2$ (i.e.) $a_1 = a_2$

Hence the mapping by g is 1-1

Step: 3 Suppose b is an even integer

let $a = b/2$ then $a \in z$ and

$g(a) = g(b/2) = 2(b/2) = b$

every element b in T has a pre image

in Z.

So mapping by g is onto.

Step: 4

let a and $b \in Z$

$g(a+b) = 2(a+b)$

$= 2a + 2b$

$= g(a) + g(b)$

Hence $(Z,+)$ & $(T,+)$ are isomorphic

Semi group.

Step: 5 Let $S = N \times N$ is being the set of positive integer and $*$ be an operation on S given by $(a,b)*(c,d) = (a+c, b+d)$

Show that S is a semigroup

Define $f:(S,*) \rightarrow (Z,+)$ by $f(a,b) = a-b$

Show that f is a isomorphism.

Sol: Let x,y,z be the ordered pairs (a,b) and (e,f) respectively in $N \times N$

Then $(xy)z = (x*y)*z$

$= [(a,b)*(c,d)]*z$

$= (a+c, b+d)*(e,f)$

$= [(a+c)+e, (b+d)+f]$

$= [a+c+e, b+d+f]$

Since $a,b,\ldots f$ are the integers

$x(yz) = x*(y*z)$

$= (a,b)*((c,d)*(e,f))$

$= (a,b)*((c+e),(d+f))$

$= (a+(c+e), b+(d+f))$

$= (a+c+e, b+d+f)$

$= (a)$

Hence $(x*y)*z = x*(y*z)$

So $*$ associative and S is a semigroup

Now, $f(x*y) = f((a,b)*(c,d))$

$= f(a+c, b+d)$ by defn of $*$.

$= (a+c)-(b+d)$ by defn of f

$= (a-b)+(c-d)$.

$= f(a,b) + f(c,d)$

$= f(x)+f(y)$

So f is a homomorphism.

Properties of homomorphisms:

Thm: 1 If g is a semigroup Isomorphism from $(S,*)$ to (T,Δ) show that g^{-1} is an Isomorphism from (T,Δ) to $(S,*)$

Pf: g is an isomorphism from $(S,*)$ to (T,Δ)

So g is 1-1 onto mapping

Hence g^{-1} exists and is 1-1 mapping from T to S

Let a' & b' be any two elem of T

Since g is onto, we can find elem a and b in S such that $g(a) = a'$ & $g(b) = b'$

Then $a = g^{-1}(a')$ and $b = g^{-1}(b')$

Now, $g^{-1}(a' \Delta b') = g^{-1}(g(a) \Delta g(b))$

$= g^{-1}(g(a * b))$ Since g is an isomorphism.

$= (g^{-1} o g)(a * b)$

$= (a * b)$

$= g^{-1}(a') * g^{-1}(b')$

Hence g^{-1} is an isomorphism.

Since g is onto. we can find elems a and b in S such that $g(a) = a'$ and $g(b) = b'$

Then $a = g^{-1}(a')$ and $b = g^{-1}(b')$

Now, $g^{-1}(a' \Delta b') = g^{-1}(g(a) \Delta g(b))$

$= g^{-1}(g(a * b))$ since g is an isomorphism.

$= (g^{-1} \circ g)(a * b)$

$= a * b$

$= g^{-1}(a') * g^{-1}(b')$

Hence g^{-1} is an isomorphism.

— x —

Thm: 2 If g is a homomorphism from a commutative semi group $(S, *)$ onto a semi group (T, Δ), then (T, Δ) is also commutative.

Pro: Let t_1 & t_2 be any two elements of T.

As g is onto, there exists element s_1 & s_2 in S such that $g(s_1) = t_1$ and $g(s_2) = t_2$

Now, $t_1 \Delta t_2 = g(s_1) \Delta g(s_2)$

$= g(s_1 * s_2)$ as

I.T g is a homorphism.

$= g(s_2 * s_1)$ as S is commulative under $*$

$= g(s_2) \Delta g(s_1)$ as g is a homomorphi

$= t_2 \Delta t_1$

Hence (T, Δ) is commulative

Thus isomorphism preserves the property of commutative

Thm: 3 The property of idempodency is preserved under a semigroup homomorphism.

Sol Let g be a semigroup homomorphism from $(S, *)$ to (T, Δ)

$$g(a*b) = g(a) \Delta g(b) \longrightarrow (1)$$

So we have $g(a*a) = g(a) \Delta g(a) \longrightarrow (2)$

Now let a be an idempotent element of S.

Then $N * a = a$ ——→ (3)

using (3) in (2) we get $g(a) = g(a) \Delta g(a)$ ——→ (4)

Eq: (4) clearly shows that $g(a)$ is an idempotent element of T.

So if a is an idempotent element in s, then its image $g(a)$ is an idempotent element in T.

<u>Thm</u>: 4

Let $(S, *)$ and (T, Δ) be monoids with identities e and e' respectively. Let $g: S \to T$ be an onto semigroup homomorphism. Then $g(e) = e'$

Pf:

Let b be any elem of T.

Since g is onto, there is an element a in S such that $g(a) = b$

Now $a = a * e$ (e being the identity element)

$b = g(a) = g(a * e)$

$= g(a) \Delta g(e)$ (homomorphism of g

$= b \Delta g(e) \longrightarrow (1)$

Again $a = e * a$

$b = g(a) = g(e * a)$

$= g(e) \Delta g(a)$ (homomorphism of g)

$= g(e) \Delta b \longrightarrow (2)$

Combining (1) & (2), we get

$b \Delta g(e) = g(e) \Delta b = b$

This shows that $g(e)$ is the identity

$\therefore g(e) = e'$

Def (Monoid homomorphism)

Let $(M, *, e_M)$ and (T, Δ, e_T) be any two monoids. A mapping $g: M \rightarrow T$ such that for any two elements $a, b \in M$. $g(a*b) = g(a) \Delta g(b)$ and $g(e_M) = e_T$ is called a monoid homomorphism.

Ex Let N be the set of an natural numbers. Define $g: N \rightarrow N$ by $g(n) = 2n$

for all n. Then g is a semigroup homomorphism from $(N, +)$ into itself

But $g(1) = 2 \neq 1$

Thm: 5

Show that the monoid homomorphism preserves the property of invertibility

Pr: Let $(M, *, e_m)$ and (T, Δ, e_T) be any two monoids and let $g: M \to T$ be a monoid homomorphism.

If $a \in M$ is invertible. Let a^{-1} be the inverse of a in M. We will now show that $g(a^{-1})$ will be an inverse of $g(a)$ in T.

$a * a^{-1} = a^{-1} * a = e_m$ (by the def of inverse)

So $g(a * a^{-1}) = g(a^{-1} * a) = g(e_m)$

Hence $g(a) \Delta g(a^{-1}) = g(a^{-1}) \Delta g(a)$

$= g(e_m)$

(since g is a homo

But $g(e_m) = e_T$ (since g is monoid homo.)

$\therefore g(a) \Delta g(a^{-1}) = g(a^{-1}) \Delta g(a)$

$= e_T$

$g(a^{-1})$ is a inverse of $g(a)$

i.e.) $g(a)$ is invertible

Thus the property of invertiblity is Preserved under monoid homomorphism.

Subsemigroups & Submonoids:-

Def:- (Subsemigroup)

let $(S, *)$ be a semigroup & T a subset of S. If the set T is closed under the operation $*$, then $(T, *)$ is said to be a subsemigroup of $(S, *)$

Def: let $(M, *)$ be a monoid with identity e and T a subset of M. If the set T is closed under the operation $*$ &.

$x^{y} \in T$, Then $(T, *)$ is said to be a submonoid of $(M, *, e)$

Ex:-

1) For the semigroup $(N, *)$ where N is the set of all natural numbers. let T be the set of multiples of a +ve integer m then $(T, *)$ is a subsemigroup of $(N, *)$

Thm:1

For any commutative monoid $(M, *)$ the set of idempotent element of M form a submonoid.

Pr

Let S be the set of idempotent elem. of M and e be the identity elem of M. As $e*e=e$ by defn, of identity, e is an idempotent elem in M & $e \in S$

We shall show that S itself is a monoid with respect to the operation $*$ on M. let $a, b \in S$, The $a*a=a$ & $b*b=b$

Now $(a*b)*(a*b) = (a*b)*(b*a)$ $(\because (M,*)$ is commut.$)$

$= a*(b*b)*a$ (asso. prop)

$= a*b*a$ (b is idempotent)

$= a*(b*a)$

$= a*(a*b)$ (commut. prop)

$= (a*a)*b$ (assoc.)

$= a * b$ (a is idemp)

$(a * b)$ is idempotent of M so $a * b \in S$

Thus $a * b \in S$ for all $a, b \in S$

So S is closed under $*$ & S is

submonoid.

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4/7/20 Date

II – Unit

Mathematical Induction:-

Principle of mathematical induction:-

Let $P(n)$ be a statement or Proposition involving the natural numbers n. Then,

(i) if $P(1)$ is true and

(ii) if $P(k+1)$ is true on the assumption that $P(k)$ is true

we conclude that a statement $P(n)$ is true for all natural numbers n.

Hence to prove that a statement $P(n)$ is true for all natural numbers we must go through two steps.

First :- we must prove that $P(1)$ is true.

Second :- Assuming that $P(k)$ is true we must p.t $P(k+1)$ is also true

The 1st step is called the basis step of the proof

The 2nd step is called the induction step of the proof

Pbm:1

Prove by induction method for $n \geq 1$ $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

Sol: Let $P(n)$ denote the statement

$P(n): 1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$

Let $n=1$, L.H.S of $P(1) = 1^2 = 1$

R.H.S of $P(1) = \frac{1(1+1)(2+1)}{6} = \frac{1 \times 2 \times 3}{6}$

$= 1$

$=$ L.H.S of $P(1)$

$P(1)$ is true.

Let us assume that $P(k)$ is true.

i.e.) $P(k): 1^2 + 2^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$ is true

we claim that $P(k+1)$ is

$1^2 + 2^2 + \dots + (k+1)^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

Now, $1^2+2^2+\dots+k^2+(k+1)^2 = (1^2+2^2+\dots+k^2)+(k+1)^2$

$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$

($\because P(k)$ is true)

$= \frac{k(k+1)(2k+1)+6(k+1)^2}{6}$

$= \frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$

$= \frac{(k+1)(2k^2+7k+6)}{6}$

$= \frac{(k+1)(k+2)(2k+3)}{6}$

$\therefore P(k+1)$ is true

Thus if $P(k)$ is true, then $P(k+1)$ is also true.

By the principle of Mathematical induction.

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ for all $n \geq 1$

9) Show that $a^n - b^n$ is divisible by $(a-b)$ for all $n \in N$

Q4

Sol:

Let $P(n)$: $a^n - b^n$ is divisible by $(a-b)$

Put $n=1$, $P(1) = a^1 - b^1$ ~~is divisible~~ $= a-b$ is divisible by $(a-b)$

Let $a^k - b^k = c(a-b)$

So, $a^k = b^k + c(a-b) \longrightarrow (1)$

Now, $a^{k+1} - b^{k+1} = a^k \cdot a - b^k \cdot b$

$= a\left(b^k + c(a-b)\right] - b^k \cdot b$

Substituting for a^k from (1).

$= ab^k + ac(a-b) - b^k \cdot b$

$= b^k(a-b) + ac(a-b)$

$= (a-b)\{b^k + ac)$

$\therefore a^{k+1} - b^{k+1}$ is divisible by $(a-b)$

(i) if $P(k)$ is true, then $P(k+1)$ is also true.

By the principle of mathematical induction $P(n)$ is true for all $n \in N$.

(6) Prove by mathematical induction that $2^n > n$ for all $n \in N$.

Sol: Let $P(n): 2^n - n > 0$

Put $n = 1$, $P(1) = 2^1 - 1 = 2 - 1 = 1 > 0$

So, $P(1)$ is true.

Let us assume that $P(k)$ is true.

Here k is a positive integer

(i) $2^k - k$ is positive

Let $2^k - k = m$ —— (i)

To prove that $P(k+1)$ also is true

Consider $2^{k+1} - (k+1)$

Now, $2^{k+1} - (k+1) = 2 \cdot 2^k - k - 1$

$= 2(k+m) - k - 1$

(Substituting for 2^k from (i))

$= k + 2m - 1$

$=$ +ve number ($\because m$ is +ve)

(ii) if $P(k)$ is true, $P(k+1)$ is also true

So by the principle of mathematical induction, $P(n)$ is true.

i.e.) $2^n - n > 0$

So, $2^n > n$ for all $n \in N$.

2b

(1) If $A_1, A_2, \dots, A_n$ are any n sets, show by mathematical induction that $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$

Sol: Let $P(n)$ be the statement that the equality holds for n sets.

Basis step: $P(1)$ is the statement that $\overline{A_1} = \overline{A_1}$ which is obviously true

Induction step: suppose $P(k)$ is true for any k sets.

i.e.) $\overline{A_1 \cup A_2 \cup \dots \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k}$

for any k sets $A_1, A_2, \dots A_k$

So let $A_1, A_2, \ldots, A_{k+1}$ be any $k+1$ sets.

Let $B = A_1 \cup A_2 \cup \cdots \cup A_k$

Then $\overline{B} = \overline{A_1 \cup A_2 \cup \cdots \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}$

L.H.S of $P(k+1) = \overline{A_1 \cup A_2 \cup \cdots \cup A_k \cup A_{k+1}}$

$= \overline{(A_1 \cup A_2 \cup \cdots \cup A_k) \cup A_{k+1}}$

(associative prty of Union)

$= \overline{B \cup A_{k+1}}$

$= \overline{B} \cap \overline{A_{k+1}}$ (by De-morgan's law for two sets)

$= (\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_k}) \cap \overline{A_{k+1}}$

$= \left(\bigcap_{j=1}^{k} \overline{A_j}\right) \cap \overline{A_{k+1}}$

$= \bigcap_{j=1}^{k+1} \overline{A_j}$ = R.H.S of $P(k+1)$

if $P(k)$ is true, $P(k+1)$ is also true

so by the Principle of mathematical induction.

$P(n)$ is true for all $n \geq 1$

Q) Suppose we have stamps of two different denominations Rs 3 and Rs 5 show that it is possible to make up exactly any postage of n Rupees, where $n \geq 8$ is an integer, using stamps of these two denominations.

Sol: Let p(n) be the statement p(n): It is possible to make up exactly a postage of Rs n using Rs 3 & Rs 5 stamps.

Clearly, P(8) is true as one Rs 3 stamps and one Rs 5 are enough to make up a postage of Rs 8.

Now, assume that p(k) is true for some $k \geq 8$. Suppose we make up a postage of Rs k using at least one Rs 5 stamp. Replacing a Rs 5 stamp by two Rs 3 stamps will yield a way to make up a postage of Rs (k+1). On the other hand, suppose we make up a postage of Rs k using Rs 3 stamps only. Since $k \geq 8$, there must be at least three Rs 3 stamps.

Replacing three Rs 3 stams by two Rs 5 stamps will yields a way to make up a postage of Rs (k+1)

Thus if P(k) is true for some k ≥ 8, then P(k+1) is also true.

Thus by the principle of mathematical induction, P(n) is true for all n ≥ 8

Exercises :- (1) using mathematical induction

P.T. $1^3+2^3+\dots+n^3 = \frac{n^2(n+1)^2}{4}$

(2) P.T using mathematical induction

$2+5+8+\dots+(3n-1) = \frac{n(3n+1)}{2}$

Recurrence Relations and Generating Functions

Recurrence :-

Recurrence is a way of giving information / instruction in terms of prior knowledge; for example, if somebody requires the route of your house, you can say "Go to a lankar theatre. Travel through the road opposite to the theatre then turn to the third left cutting my house in the second on the left side". Of course this instruction is useful only when one know how to go to Alankar theatre. Even if one does not know, he can know the way to reach the theatre from others & thus the instruction is based on some prior knowledge information. this prior process is called recursion.

Ex:1 The Fibonacci numbers can be defined as follows.

$F_0 = 1, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2}$

Ex:2 ^nC_r can be defined as follows.

$^nC_0 = 1, \quad ^nC_n = 1$

$^nC_r = {}^{(n-1)}C_r + {}^{(n-1)}C_{r-1}, \quad n > r > 0.$

Ex:
3d A Ackermann's function can be defined as follows

$A(0,y) = y+1$

$A(x+1, 0) = A(x,1)$

$A(x+1, y+1) = A(x, A(x+1, y))$

Recursion, Iteration and Induction.

while we calculate function value through recursion, we start with the required expression in terms of earlier values. we have backwards untill we reach the basis. and in Iteration (or iterative procedure). we start with the basis and work forward using relation and stop when the required value is known.

Example: 4 Calculate F_4 of the Fibonacci numbers using (i) recursion (ii) Iteration

Ans: (i) $F_4 = F_3 + F_2 = (F_1 + F_0) + F_2$

$= (F_1 + (F_0 + F_1)) + F_0 + F_1$

$= (1 + (1+1)) + 1 + 1$

$= 5$

(ii) $F_2 = F_0 + F_1 = 1+1 = 2$

$F_3 = F_1 + F_2 = 1+2 = 3$

$F_4 = F_2 + F_3 = 2+3 = 6$

—— x ——

Recursion and iteration:-

1) Usually iterative computations are faster than recursive computations.

2) But recursive definition give more insight into the interpretation of the given fn!

3) A recursive programme is more difficult to dbug that a corresponding ittative programme.

4) In general there are many problems involving recursion for which iterative sol. either do not oxist or not easily fond.

Recursion and induction:-

Induction is used for proving properties expressed as a fact of natural number. we can also define a property or a set having a property by induction. the corresponding defn: is called a induction defn:

An inductive definition of a property or a set P is given as follows.

1) Given a finite set A whose elements have the property P.

2) The elements of a set B, all of which are constructed from A where the property P.

3) The elements constructed as in (1) and (2) are the only elements satisfying property P. so, is inductive defn: we use recursive defn: in the forward direction.

Also, more that proof by induction can be used whenever recursive is used.

Programming language & recursion:

A procedure or subroutine is a tool in a programming language which enable a programme to express just one on algorithm. which is used in many place while executing the programme.

A procedure that contain a procedure call to itself is know as a recursive procedure. Recursive procedure are applicable in most of the programming language like ALGOL etc and some recent version of FORTRAN.

In some very early version of FORTRAN. recursive procedure were not allowed.

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when a recursive procedure is to be implemented using a computer language

1) each time a procedure calls itself it must be number to a arg:

2) There must be a decision criterion for stopping the computation.

---x---

Def: (Recursive definition of a polynomial)

The set $S[x]$ of all polynomials who co-efficient are elements of S is defined as follows.

1. Any element of S is a polynomial of degree zero.

2. $p(x) x + a$ is a polynomial of degree n when $p(x)$ is a polynomial of degree $n-1$ & $a \in S$

3. Only those expression obtained by using (1) & (2) finite number of times are polyn:

Ex(ii): 39

Consider $F(x) = 5x^3 + 4x^2 + 3x + 12$

This can be defined using recursive defn. as follows

$$F(x) = ((((5)x+4)x+3)x+12)$$

Note: A polyn. defn. recursively is said to be in telescopic form. The method of writing a polyn. in recursive form (telescoping form) is called Horner's method.

Ex(i):

1) Let $p(x) = x^5 + 5x^4 - 15x^3 + x - 10$ in telescopic form.

Ans:

$$p(x) = ((((1)x+5)x-15)x+0)x+1)x-10$$

Ex: A^b

use Horner's method to write $p(x) = x^4 + 2x^3 + 3x^2 + 4x$ in telescoping form. Also mention the number of multiplication and addition. Subtraction involved in telescopic form, compare it with usual defn!

Sol:

$p(x) = (((1)x + 2)x + 3)x + 4)x + 0$

we require 4 multiplication and 4 additions

In the usual form we require 7 multiplication and 4 addition

(for example)

$p(2) = 1(2^4) + 2(2^3) + 3(2^2) + 4(2) + 0$

we multiply 4(2) (once), 2^2 (once) 3(2^2) (once), $2^3 = 2(2^2)$ (once), 2(2^3) (once), 1(2^4) (once)

Thus we require 7 multiplication

Thus by writing a polynomial in te[lescopic] form the number of multiplication [is] reduced form 7 to 4.

Def:- A sequence of integers (also called a discrete function) is a function from N into Z (where N is the set of all natural numbers and Z is the set of all integers).

Ex:- i) The Fibonacci numbers $F_0 = 1$, $F_1 = 1$, $F_2 = 2$ ...... is a sequence of integers.

Def: Let S be a sequence of integers. A recurrence relation on S is a formula that relates all but a finite number of terms of S to previous terms of S.

(i) there exists a k_0 in the domain of S such that S(k), for $k > k_0$, is expressed in terms of some of the terms of the sequence preceding S(k). The terms not defined by the formula are said to form the initial conditions (or boundary conditions, or basis) of the sequence.

Ex (1) Find the recurrence relation and basis for the sequence $(1, 3, 3^2, \ldots)$

Sol: Take $(0, 1, 2, \ldots)$ as the domain of the sequence. Then $a_0 = 1$, $a_1 = 3$, $a_2 = 3^2$, Hence $a_n = 3a_{n-1}$ is the recurrence relation. $a_0 = 1$ is the basis.

Ex (2) Consider D, defined by $D(k) = 5 \cdot 2^k$, $k \geq 0$. Find the recurrence relation on D.

Sol: For $k \geq 1$, $D(k) = 5 \cdot 2^k$ and

$$D(k-1) = 5 \cdot 2^{k-1}$$

So, $\dfrac{D(k)}{D(k-1)} = 2$

Hence the recurrence relation is

$$D(k) - 2D(k-1) = 0, \quad k \geq 1$$

The initial condition $D(0) = 5$

Def: A recurrence relation on a sequence S is of order k if $T(n)$ is expressed as a functions of $T(n-1), \ldots, T(n-k)$ and $T(n-k)$ appear in the function.

Ex:

The relation $T(n) = 2(T(n-1))^2 - nT(n-3)$ is a sequence relation of order 3.

Def:-

A recurrence relation on a sequence S is a linear recurrence relation with constant co-efficients if it is of the form.

$$S(k) + C_1 S(k-1) + \ldots + C_n S(k-n) = f(k), \quad k \geq n$$

where $C_1, C_2, \ldots C_n$ are numbers and f is a function defined for $k \geq n$. Also if $C_n \neq 0$ then the relation is said to be order n.

Note:

A linear recurrence relation with constant co-efficient is simply called a linear relation.

Df: An n^{th} order linear relation is a homogeneous relation if $f(k) = 0 \; \forall k$.

Def:- For a recurrence relation
$S(k) + C_1 S(k-1) + \dots + C_n S(k-n) = f(k)$
the associated homogeneous relation
is $S(k) + C_1 S(k-1) + \dots + C_n S(k-n) = 0$

— x —

Example:- Consider the recurrence relation

(i) $D(k) - 8D(k-1) = 0$ (ii) $C(k) - 5C(k-1) + 6C(k-2) = 2k-1$

(iii) $S(k) - 4S(k-1) - 11S(k-2)$

(iv) $T(k) - T([k/2]) + 5,\ k \geq 0$ where $[k/2]$ is the integral part of $k/2$

(i) is a homogeneous relation of order 1

(ii) is a linear relation of order 2 but not homogeneous.

(iii) is a linear non-homogeneous relation of order 2

(iv) is a recurrence relation of infinite order. for given any positive integer n

we can find k such that $T(k) = T(k-n)$

is just take $k = 2n$, then $T(2n) = T([\frac{2n}{2}])+$
$5 = T(n) + 5 = T(2n-n) + 5$.

So it is not possible to find a

fixed positive integer n such that a relation of the type,

$$T(k) + C_1 T(k-1) + \cdots + C_n T(k-n) = f(k)$$

holds for all $k \geq n$

Prob ① Find the recurrence relation satisfying $y_n = A(3)^n + B(-4)^n$

Sol

$y_n = A \cdot 3^n + B(-4)^n \longrightarrow ①$

$y_{n-1} = A\, 3^{n-1} + B(-4)^{n-1} \longrightarrow ②$

$3y_{n-1} = A \cdot 3^n + B(3)(-4)^{n-1} \longrightarrow ③$

(1)-(3) gives $y_n - 3y_{n-1} = B(-4)^{n-1}(-7) \longrightarrow ④$

From ④ $y_{n-1} - 3y_{n-2} = B(-4)^{n-2}(-7) \longrightarrow ⑤$

From ④ & ⑤ $y_n - 3y_{n-1} = -4(y_{n-1} - 3y_{n-2})$

i.e.) $y_n + y_{n-1} - 12y_{n-2} = 0$

Which is the required recurrence relation.

Recurrence Relation

Prob 2 46

Find the recurrence relation satisfy

(2) $y_n = (A+Bn)4^n$

Sol:

$$y_n = A\cdot 4^n + Bn\,4^n \longrightarrow (1)$$

$$y_{n-1} = A\cdot 4^{n-1} + B(n-1)4^{n-1} \longrightarrow (2)$$

From (2), $4y_{n-1} = A\cdot 4^n + B(n-1)\cdot 4^n \longrightarrow (3)$

(1) - (3) gives $y_n - 4y_{n-1} = B\cdot 4^n \longrightarrow (4)$

Using (4), $y_{n-1} - 4y_{n-2} = B\cdot 4^{n-1} \longrightarrow (5)$

From (4) & (5) we get

$$y_n - 4y_{n-1} = 4(y_{n-1} - 4y_{n-2})$$

So that the required recurrence relation is $y_n - 8y_{n-1} + 16y_{n-2} = 0$.

Prob 3

For the sequence defined by $A(k) = k^2 - k$, $k \geq 0$ obtain the recurrence relation if A is a sequence of integers.

Sol:

$$A(k) = k^2 - k$$

$$A(k-1) = (k-1)^2 - (k-1)$$

Hence, $A(k) - A(k-1) = k^2 - (k-1)^2 - (k-(k-1))$

$|||^{ly}$, $A(k-1) - A(k-2) = 2(k-1) - 2 = 2k$

So, $(A(k) - A(k-1)) - (A(k-1) - A(k-2)) = 2$

i.e.) $A(k) - 2A(k-1) + A(k-2) - 2 = 0$

is the recurrence relation for the

sequence $A(k)$.

(7) Solution of Finite ORDER HOMOGENEOUS (LINEAR) RELATION.

Suppose we want to find $S(k)$, wher[e] the recurrence relation on S is

$S(k) - 7S(k-1) + 10\,S(k-2) = 0$, $S(0) = S(1) = 4$

To find $S(2)$ we can use $S(2) - 7S(1) + 10S(0) = 0$ giving the value -20. Hence $S(2) = -2[0]$ This process can be repead for finding $S(k)$ for $k > 0$. But this process is tedious and time-consuming for large values of k. If $S(k)$ is got as a fun' of k, then $S(k)$ can be directly evaluated. Such a function is called a closed from expression for a sequence S.

Ex:1 Find a closed form expression for the recurrence relation

$D(k) - 2D(k-1) = 0$, $D(0) = 5$

Sol: $D(k) = 2D(k-1)$, As $D(0) = 5$, $D(1) = 2D(0) = 2(5)$. We can prove by induction that $D(k) = 5 \cdot 2^k \ \forall\ k \geq 0$

Hence $D(k) = 5 \cdot 2^k$ is a closed from expression for D.

It is not possible to solve all recurrence relations. Also there is no single algorithm to solve the relations that are solvable. In any case there are simple algorithms to solve n^{th} order linear recurrence relations. First of all we give an algorithm for solving n^{th} order homogeneous (linear) eq:. For this, we require the definition of the characteristic equation of a homogeneous relation.

Def: The characteristic eq: of the homogeneous relation of order n

$$S(k) + C_1 S(k-1) + \dots + C_n S(k-n) = 0$$

is the n^{th} degree equation

$$a^n + C_1 a^{n-1} + C_2 a^{n-2} + \dots + C_{n-1} a + C_n = 0$$

The left hand side of this eq: is called the characteristic polynomial

Ex: Find the characterization eq: of $J(k) - 4J(k-1) + 4J(k-2) = 0$

Sol: the charact. eq: is

$$a^2 - 4a + 4 = 0$$

Algorithm for solving r^{th} order Homogeneous Recurrence Relation:

Step: 1 - Write the characteristic equation of the given homogeneous relation.

Step 2 - Find all the roots of the characteristic equation. (They are called characteristic roots)

Step 3 :- (i) If the roots $a_1, a_2, \ldots a_n$ are distinct then the general sol of the recurrence relation is $s(k) = b_1 a_1^k + b_2 a_2^k + \ldots + b_n a_n^k$ ——→ (1)

(ii) If the root a_j is repeated p times

$b_j a_j^k$ is replaced by $(c_0 + c_1 k + \ldots + c_{p-1} k^{p-1}) a_j^k$ [illegible]

$(c_0 + c_1 k + \ldots + c_{p-1} k^{p-1}) a_j^k$

(In particular, if is a double root then $b_j a_j^k$ is replaced by $((c_0 + c_1 k) a_j^k)$)

Step 4 :- If n initial condition are given, obtain n linear equations in n unknowns $b_1, b_2, \ldots b_n$ (got in (1)) by replacing L.H.S of (1) by the given values. If possible solve these eq^n.

Note:- We have a general method for solving quadratic eq^n. But is difficult to solve eq^n of higher degree. These eq^n can be solved by removing linear factors (got by trial and error). The following rule will be useful in some cases.

If a characteristic polynomial has integral roots then the roots will be factors of the independent term of the polynomial. In such cases the trial and error method can be applied to factors of the independent term first.

Ex: 1 Solve the following recurrence relation.

$S(k) - 10S(k-1) + 9S(k-2) = 0$, $S(0) = 8$, $S(1) = 11$

Sol: Step 1: The characteristic equation is

$$a^2 - 10a + 9 = 0$$

Step 2: The roots are 1, 9 (This can be got by factorisation of $a^2 - 10a + 9$ or using the formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ for roots of a quadratic eqn)

Step 3:- As the root are distinct.

$$S(k) = b_1 \cdot 1^k + b_2 \cdot 9^k = b_1 + b_2 9^k$$

Step 4: $9 = s(0) = b_1 + b_2 \cdot 4^0 = b_1 + b_2$

$11 = s(1) = b_1 + b_2 \cdot 4^1 = b_1 + 4b_2$

Solving these eq' we get $b_2 = \frac{1}{2}$, $b_1 = 2$

Hence $s(k) = 8 + 4^k$

__Worked Ex: 1__

Solve $D(k) - 8D(k-1) + 16D(k-2) = 0$ where $D(2) = 16$, $D(3) = 80$

Sol: The char eq' is

$$a^2 - 8a + 16 = 0$$

The roots are 4, 4. As the roots are repeated,

$$D(k) = (C_0 + C_1 k) 4^k$$

$16 = D(2) = (C_0 + 2C_1)4^2 = 16(C_0 + 2C_1) = 16C_0 + 32C_1$

$80 = D(3) = (C_0 + 3C_1)4^3 = 64C_0 + 192C_1$

That is $64C_0 + 192C_1 = 80 \longrightarrow (1)$

$16C_0 + 32C_1 = 16 \longrightarrow (2)$

$(1) - 4 \times (2)$ gives $(192 - 128)C_1 = 80 - 64$

So $C_1 = \frac{1}{4}$, $C_0 = 1 - 2C_1 = \frac{1}{2}$

Hence $D(k) = (\frac{1}{2} + \frac{1}{4}k)4^k$

2) Find $f(n)$ if $f(n) = 7f(n-1) - 10f(n-2)$ given that $f(0) = 4$ & $f(1) = 17$.

Sol: The relation is

$f(n) - 7f(n-1) + 10f(n-2) = 0$

Hence the charn eqn is

$a^2 - 7a + 10 = 0$

The roots are 2, 5

So, $f(n) = b_1 2^n + b_2 \cdot 5^n$

$4 = f(0) = b_1 2^0 + b_2 5^0 = b_1 + b_2$

$17 = f(1) = b_1 2^1 + b_2 5^1 = 2b_1 + 5b_2$

Solving the eqn:

$b_1 + b_2 = 4$

$2b_1 + 5b_2 = 17$

we get $b_2 = 3$, $b_1 = 1$

Hence $f(n) = 1 \cdot 2^n + 3 \cdot 5^n = 2^n + 3 \cdot 5^n$

3) Solve the recurrence relation

$S(k) - 4S(k-1) - 11S(k-2) + 30S(k-3) = 0$

$S(0) = 0$, $S(1) = -35$, $S(2) = -85$

Sol: The charn eqn is

$a^3 - 4a^2 - 11a + 30 = 0$

As it is a cubic eq^n, we need to separate a linear factor first. We check whether the factors of 30 i.e. $\pm 1, \pm 2, \pm 3, \pm 5, \pm 10, \pm 15$ are roots.

± 1 are not roots of the characteristic eq^n. But $a^3 - 4a^2 - 11a + 30 = 0$ is satisfied when $a = 2$.

Hence 2 is a root and $a - 2$ is a factor of the characteristic polynomial $a^3 - 4a^2 - 11a + 30$. By synthetic division

$$\begin{array}{c|cccc} 2 & 1 & -4 & -11 & 30 \\ & & 2 & -4 & -30 \\ \hline & 1 & -2 & -15 & 0 \end{array}$$

$a^3 - 4a^2 - 11a + 30$.

So $a^2 - 2a - 15$ is a factor.

Roots of $a^2 - 2a - 15 = 0$ are -3 and 5.

Hence the roots of the char. eq^n are $2, -3$, and 5.

Thus $S(k) = b_1 2^k + b_2 (-3)^k + b_3 5^k$

As $S(0) = 0$, $S(1) = -35$ & $S(2) = -85$,

we get

$0 = S(0) = b_1 2^0 + b_2 (-3)^0 + b_3 (5)^0 = b_1 + b_2 + b_3$

$-35 = S(1) = b_1 2^1 + b_2 (-3)^1 + b_3 (5)^1 = 2b_1 - 3b_2 + 5b_3$

$-85 = S(2) = b_1 2^2 + b_2 (-3)^2 + b_3 (5)^2 = 4b_1 + 9b_2 + 25b_3$

So we've to solve the eqn

$b_1 + b_2 + b_3 = 0 \quad \ldots (1)$

$2b_1 + 3b_2 + 5b_3 = -35 \quad \ldots (2)$

$4b_1 + 9b_2 + 25b_3 = -85 \quad \ldots (3)$

$2\times(1) - 2$ gives $5b_2 - 2b_3 = 25 \quad \ldots (4)$

$2\times(2) - 3$ gives $15b_2 - 15b_3 = 15$

that is $b_2 + b_3 = -1$

using (1) we get $b_1 = 1$

So $S(k) = 1\cdot 2^k + 4\cdot(-3)^k - 5\cdot 5^k$.

4) Write the recurrence relation for Fibonacci numbers and solve it

Ans) The recurrence relation is

$F(n) - F(n-1) - F(n-2) = 0$

The characteristic eqn is

$a^2 - a - 1 = 0$

The roots are $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$

Hence $F(n) = b_1\left(\frac{1+\sqrt{5}}{2}\right)^n + b_2\left(\frac{1-\sqrt{5}}{2}\right)^n \longrightarrow (1)$

Also $1 = F(0) = b_1 + b_2$

$1 = F(1) = b_1\left(\frac{1+\sqrt{5}}{2}\right) + b_2\left(\frac{1-\sqrt{5}}{2}\right) \longrightarrow (2)$

As $b_2 = 1 - b_1$, (3) can be written as

$$b_1(1+\sqrt{5}) + (1-\sqrt{5})(1-b_1) = 2$$

i.e. $(1+\sqrt{5}-1+\sqrt{5})\,b_1 = 1+\sqrt{5}$

So $b_1 = \frac{\sqrt{5}+1}{2\sqrt{5}}$ and $b_2 = 1 - \frac{\sqrt{5}+1}{2\sqrt{5}} = \frac{\sqrt{5}-1}{2\sqrt{5}}$

Hence the recurrence relation for the Fibonacci sequence is

$$F(n) = \left(\frac{\sqrt{5}+1}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{\sqrt{5}-1}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

Finished II unit

Unit-3

Solution of non-homogeneous Relation:

In this case of non homogeneous recurrence relation the general soln is the sum of

(i) Soln for the corresponding homogeneous relation.

(ii) particular soln depending on the R.H.S of the given recur. relat. it can be found as in the previous section.

For finding the particular soln we adopt the following produce.

2 m

Procedure for finding the particular soln:-

Step 1 :- a) If the R.H.S of the recurrence relation is a $a_0 + a_1 k + \dots + a_m k^m$ is place substitute $d_0 + d_1 k + \dots + d_m k^m$ in place of $T(k)$, $d_0 + d_1(k-1) + \dots + d_m (k-1)^m$ in place of $T(k-1)$ etc in the g.r. relation.

Step 2:- At the end of step 1 we get a polynomial in k with coeffi: $d_0, d_1, \ldots$ on L.H.S which is equal to the R.H.S of the given recurrence relation. Equate the coeffi: of powers of k on both sides to get values $d_0, d_1 \ldots$

Step 3:- The general sol: is the sum of the sol: for the homogen: relation and the particular sol: got in step 2. Use initial conditions for getting the values of unknowns (b_1, b_2 etc)

Note

we discuss some particular cases now

1. If the R.H.S of the given recurrence relation is a constant a_0, then replace $T(k), T(k-1) \ldots$ by d_0.

2. If the R.H.S is ca^k and a coincides with a characteristic root. The above method of

3. If the R.H.S is $a_0 + a_1 k$, replace $T(k)$ by $d_0 + d_1 k$, $T(k-1)$ by $d_0 + d_1(k-1)$ etc.

3. When the R.H.S is ca^k and a coincides with a charp root, the above method fails; when a is a simple root of the charcl eqn take $d_0 k a^k$ (see worked Ex: 2). When a is a double root of the charcl eqn take $d_0 k^2 a^k$ (see worked Ex: 5).

<u>Ex: 1</u> Solve $T(k) - 7T(k-1) + 10T(k-2) = 6 + 8k$ with $T(0) = 1$ and $T(1) = 2$.

Soln:- a) Homogeneous Soln:-

The characteristic eqn is $a^2 - 7a + 10 = 0$

roots are 2, 5

Hence the homogl Soln is

$$b_1 2^k + b_2 5^k$$

b) Particular Soln:-

Step 1: As the R.H.S of the given relation is $6 + 8k$ take $d_0 + d_1 k$ replace $T(k)$, $T(k-1)$, $T(k-2)$ by $d_0 + d_1 k$, $d_0 + d_1(k-1)$, $d_0 + d_1(k-2)$ respectively.

Step 2: we get

$d_0 + d_1 k - 7(d_0 + d_1(k-1)) + 10(d_0 + d_1(k-2))$

$= 6 + 8k$

That is $(4d_0 - 13d_1) + (4d_1)k = 6 + 8k$

Equating the corresp. coeff. we get

$4d_0 - 13d_1 = 6$ and $4d_1 = 8$

Solving $d_1 = 2$, $d_0 = 8$

Hence the particular sol. is $8 + 2k$

and the general sol. is

$T(k) = b_1 2^k + b_2 5^k + 8 + 2k$

Now,

$1 = T(0) = b_1 2^0 + b_2 5^0 + 8 + 0 = b_1 + b_2 + 8$

$2 = T(1) = b_1 2^1 + b_2 5^1 + 8 + 2 \quad) = 2b_1 + 5b_2 + 10$

Solving $b_1 + b_2 + 8 = 1$, $2b_1 + 5b_2 + 10 = 2$

we get $b_1 = -9$, $b_2 = 2$

$\therefore T(k) = -9 \cdot 2^k + 2 \cdot 5^k + 8 + 2k$

W.E 1 (9th Exam)
Solve $S(k) - S(k-1) - 6S(k-2) = -30$ when
$S(0) = 20$, $S(1) = -5$.

Sol: a) Homogeneous Sol:

The charact. eqn is $a^2 - a - 6 = 0$

Roots are $-2, 3$

Hence the hom. soln is

$b_1(-2)^k + b_2 3^k$

b) Part. Soln:-

As the R.H.S of the recurrence relation is a constant, take d. Replace $S(k)$, $S(k-1)$, $S(k-2)$ by d. we get

$d - d - 6d = -30$ (i.e.) $d = 5$

Hence the particular soln is 5

The general soln is

$S(k) = b_1(-2)^k + b_2 3^k + 5$

$20 = S(0) = b_1 + b_2 + 5$, (i.e.) $b_1 + b_2 = 15$

$-5 = S(1) = b_1(-2) + b_2 3 + 5$

(i.e.) $-2b_1 + 3b_2 = -10$

Solving $b_1 + b_2 = 15$, $-2b_1 + 3b_2 = -10$

we get $b_1 = 11$, $b_2 = 4$

Hence the soln is

$S(k) = 11(-2)^k + 4 \cdot 3^k + 5$

W.k: 2 Solve $S(k) - 3S(k-1) - 4S(k-2) = 4^k$

Sol: a) Homogen. Sol:

The Charc. eqn is $a^2 - 3a - 4 = 0$

The roots are $-1, 4$

Hence the homo. sol: is

$b_1(-1)^k + b_2 4^k$

b) Particular sol:

The R.H.S is 4^k. The base of R.H.S is 4. As 4 is a (Simple) root of the Charc. eqn. Take $dk4^k$ (If 4 is not a root of the Charc. eqn it is enough to take $d \cdot 4^k$) Substituting $dk4^k$ for $S(k)$, $d(k-1)4^{k-1}$ for $S(k-1)$ and $d(k-2)4^{k-2}$ for $S(k-2)$ in the Recur. relation, we get

$dk4^k - 3d(k-1)4^{k-1} - 4d(k-2)4^{k-2} = 4^k$

Cancelling 4^{k-2} on both sides we get

$16dk - 12d(k-1) - 4d(k-2) = 16$

(i) $20d = 16$ $d = 0.8$

Hence the particular sol: is $(0.8)\,k\,4^k$

The general sol: is

$$s(k) = b_1(-1)^k + b_2 4^k + 0.8\,k\,4^k$$

Ex: 8

Solve $s(k) - 4s(k-1) + 4s(k-2) = 8k + 2^k$,

$s(0) = 1,\ s(1) = 1$

Sol:

a) Homog: Sol:

The charac: equ: is $a^2 - 4a + 4 = 0$

The roots are 2, 2

Hence the homog: sol: is

$(c_0 + c_1 k)2^k$

b) Particular solution is for $8k$,

Take $d_0 + d_1 k$. Replace

$s(k)$ by $d_0 + d_1 k$, $s(k-1)$ by $d_0 + d_1(k-1)$

and $s(k-2)$ in $d_0 + d_1(k-2)$ we get

$d_0 + d_1 k - 4[d_0 + d_1(k-1)] + 4[d_0 + d_1(k-2)] = 8k$

ie) $d_0 - 4d_1 + [d_1 - 4d_1 + 4d_1]k = 8k$

So $d_0 - 4d_1 = 0$ and $d_1 = 8$, $d_0 = 32$

Partl: sol: corresp: to $8k$ is $32 + 8k$.

c) Particular eqn corresponding to 2^k

Take $dk^2 2^k$ (since the base of the RHS (a) 2 is a double root of the char. eqn)

Replacing $S(k)$ by $dk^2 2^k$ etc.

$dk^2 2^k - 4[d(k-1)^2 2^{k-1}] + 4[d(k-2)^2 2^{k-2}] = 2^k$

cancelling 2^k on both sides we get

$dk^2 - 2d(k-1) + d(k-2)^2 = 1$

Simplify: $2d = 1$ (i) $d = 1/2$

particular eqn corresp. to 2^k is $(1/2) k^2 2^k$

Hence the general soln is

$S(k) = (C_0 + C_1 k)2^k + 12 + 3k + (1/2)k^2 2^k$

(1) $S(k) = 12 + 3k + (C_0 + C_1 k + k^2(1/2))2^k$

$1 = S(0) = 12 + C_0 \quad C_0 = -11$

$1 = S(1) = 12 + 3 + (C_0 + C_1 + 1/2)2$

(2) $2C_0 + 2C_1 + 16 = 1 \cdot C_1 = 7/2$

Thus $S(k) = 12 + 3k + (-11 + (7/2)k + k^2(1/2))2^k$

(3) $S(k) = 12 + 3k + (k^2 + 7k - 22)2^{k-1}$

Note: We have evalued the particular sol for 3k and 2^k seperatly we can also take $d_0+d_1k+d_2k2^k$ and find the particular sol for $3k+2^k$ in a single step.

Generating fun:

In this section we define the generating function of a sequence, we use the generating function for solving a recurrence relation.

Def:- The generating fun: of a sequence $S_0, S_1, S_2 \ldots$ is the Power series

$$G(S:z) = S_0 + S_1 z + S_2 z^2 + \ldots$$

Ex: Find the generating fun of the recurrence relation $S(k) = 2S(k-1)$, $S(0)=1$

Sol: Let $G(S:z)$ be the generating induction of the sequence $\{S(k)\}$

Then $G(S:z) = S(0) + S(1) + S(2) z^2 + \ldots$

$= S(0) + 2S(0)x + 2^2 S(0)x^2 + \dots$

$= 1 + 2x + 2^2x^2 + \dots$

$= \frac{1}{1-2x}$ $\left[\text{Since } \frac{1}{1-x} = 1 + x + x^2 + \dots\right]$

Procedure for finding the generating fn of a given recurrence relation:

Step 1: Write the given recurrence relation as an eqn with 0 on R.H.S.

Step 2: Multiply the L.H.S of the eqn got in step 1 by z^n and take the sum over all n. The resulting power series is now rewritten in terms of $G(S, z)$.

Step 3: Write $G(S, z)$ as a function of z. This is the required generating fn.

Ex: 2 Find the generating fn of Fibonacci series.

Ans: The Fibonacci series is given by

$F(n) = F(n-1) + F(n-2), n \geq 2, F(0) = F(1) = 1$

Let the generating fn be $G(S, x)$

write the recurrence relation as

$$F(n) - F(n-1) - F(n-2) = 0, \; n \geq 2$$

Hence

$$0 = \sum_{n=2}^{\infty} \left[F(n) - F(n-1) - F(n-2) \right] z^{n}, \quad \underline{n \geq 2}$$

$$= \sum_{n=2}^{\infty} F(n) z^{n} - \sum_{n=2}^{\infty} F(n-1) z^{n} - \sum_{n=2}^{\infty} F(n-2) z^{n}$$

$$= \left[\sum_{n=0}^{\infty} F(n) z^{n} - F(0) - F(1) z \right] - z \sum_{n=2}^{\infty} F(n-1) z^{n-1} - z^{2} \sum_{n=2}^{\infty} F(n-2) z^{n-2}$$

$$= \left[G(F; z) - 1 - z \right] - z \left[F(1) z + F(2) z^{2} + \cdots \right] - z^{2} \left[F(0) z^{0} + F(1) z + F(2) z^{2} + \cdots \right]$$

$$= \left[G(F; z) - 1 - z \right] - z \left[F(0) + F(1) z + F(2) z^{2} + \cdots - F(0) \right] - z^{2} \left[\sum_{n=0}^{\infty} F(n) z^{n} \right]$$

$$= G(F; z) - 1 - z - z \left[G(F; z) - 1 \right] - z^{2} G(F; z)$$

$$= (1 - z - z^{2}) G(F; z) - 1$$

That is $(1 - z - z^{2}) G(F; z) = 1$

Hence $G(F, z) = \dfrac{1}{1 - z - z^{2}}$

W.E. 1.1 Find the generating function for the following sequences (i) $S(n) = ba^n$ (ii) $S(n) = n$ (iii) $S(n) = bna^n$ (iv) $S(n) = \frac{1}{n!}$ (v) $S(n) = \begin{cases} [illegible] & \text{for } 0 \le r \le n \\ 0 & \text{for } r > n \end{cases}$

Sol:

(i) $G(S;z) = \sum_{n=0}^{\infty} ba^n z^n = b \sum_{n=0}^{\infty} (az)^n = \frac{b}{1-az}$

(ii) $G(S;z) = \sum_{n=0}^{\infty} n z^n = z + 2z^2 + 3z^3 + \dots = z(1+2z+3z^2+\dots) = \frac{z}{(1-z)^2}$

(iii) $G(S;z) = \sum_{n=0}^{\infty} bna^n z^n = baz + 2ba^2z^2 + 3ba^3z^3 + \dots$

$= abz(1+2az+3(az)^2+\dots)$

$= \frac{abz}{(1-az)^2}$

(iv) $G(S;z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots = e^z$

(v) $G(S;z) = \sum_{n=0}^{n} {}^nC_r z^r = (1+z)^n$

W.E. 1.2 If $P(n) - 6P(n-1) + 5P(n-2) = 0$, $P(0) = 2$, $P(1) = [illegible]$

What is the generating function of P^n

Sol:

Let $G(P;z)$ be the generating function for P then $G(P;z) = \sum_{n=0}^{\infty} P(n) z^n$

We replace the running index k by n.

The given relation is

$$P(n) - 6P(n-1) + 5P(n-2) = 0, \quad n \geq 2$$

Hence

$$\sum_{n=2}^{\infty} [P(n) - 6P(n-1) + 5P(n-2)] z^n = 0$$

That is $$0 = \sum_{n=2}^{\infty} P(n) z^n - 6 \sum_{n=2}^{\infty} P(n-1) z^n + 5 \sum_{n=2}^{\infty} P(n-2) z^n$$

$$= \left[\sum_{n=0}^{\infty} P(n) z^n - P(0) - P(1) z\right] - 6z \left[\sum_{n=2}^{\infty} P(n-1) z^{n-1}\right]$$

$$+ 5z^2 \left[\sum_{n=2}^{\infty} P(n-2) z^{n-2}\right]$$

$$= G(P;z) - 2 - 2z - 6z[P(1) z + P(2) z^2 + \ldots] + 5z^2 [P(0) z^0 + P(1) z + P(2) z^2 + \ldots]$$

$$= G(P;z) - 2 - 2z - 6z[G(P;z) - P(0)] + 5z^2 G(P;z)$$

$$= G(P;z) - 2 - 2z - 6z\, G(P;z) + 12z + 5z^2 G(P;z)$$

So, $$(1 - 6z + 5z^2)\, G(P;z) = 2 - 10z$$

Hence $$G(P;z) = \frac{2 - 10z}{1 - 6z + 5z^2}$$

W.E. 3:- Using the generating function solve the difference eqn: $y_{n+2} - y_{n+1} - 6y_n = 0$ given $y_1 = 1$, $y_0 = 2$.

Sol:-

Let $G(y; z)$ be the generating fn. of the sequence (y_n).

Then $G(y; z) = \sum_{0}^{\infty} y_n z^n$

As $y_{n+2} - y_{n+1} - 6y_n = 0$, $n \geq 0$

$$\sum_{n=0}^{\infty} (y_{n+2} - y_{n+1} - 6y_n) z^n = 0$$

Hence $$0 = \sum_{n=0}^{\infty} y_{n+2} z^n - \sum_{n=0}^{\infty} y_{n+1} z^n - 6 \sum_{n=0}^{\infty} y_n z^n$$

$$= \frac{1}{z^2} \left[\sum_{n=0}^{\infty} y_{n+2} z^{n+2} \right] - \frac{1}{z} \left[\sum_{n=0}^{\infty} y_{n+1} z^{n+1} \right]$$

$$- 6\, G(y; z)$$

$$= \frac{1}{z^2} \left[y_2 z^2 + y_3 z^3 + \dots \right] - \frac{1}{z} \left[y_1 z + y_2 z^2 + \dots \right]$$

$$- 6\, G(y; z)$$

$$= \frac{1}{z^2} \left[G(y; z) - y_0 - y_1 z \right] - \frac{1}{z} \left[G(y; z) - y_0 \right]$$

$$- 6\, G(y; z)$$

$$= \frac{1}{z^2}\left[G(y;z) - 2 - z\right] - \frac{1}{z}\left[G(y;z) - 2\right]$$

$$- 6G(y;z)$$

Multiplying by z^2, we get

$$G(y;z) - 2 - z - zG(y;z) + 2z - 6z^2 G(y;z) = 0$$

So, $(1 - z - 6z^2)\, G(y;z) = 2 - z$

Hence $G(y,z) = \dfrac{2 - z}{1 - z - 6z^2}$

To solve the diff: eq: resolve $G(y;z)$ into partial factions

$$1 - z - 6z^2 = (1 + 3z)(1 - 2z) \quad \text{—— (1)}$$

Let $\dfrac{2 - z}{1 - z - 6z^2} = \dfrac{A}{1 - 3z} + \dfrac{B}{1 + 2z}$

Multiplying (1) by $1 - z - 6z^2$ on both side

$$2 - z = A(1 + 2z) + B(1 - 3z) \longrightarrow (2)$$

Putting $z = -\frac{1}{2}$ in (2), we get $B = 1$

putting $z = \frac{1}{3}$ in (2) we get $A = 1$

Hence $G(y,z) = \dfrac{1}{1 - 3z} + \dfrac{1}{1 + 2z}$

y_n = coefficient of z^n in the expansion of

$(1-3z)^{-1} + (1+2z)^{-1}$

$= 3^n + (-1)^n 2^n$

W.E. 4

Solve the recurrence relation $S(n) = S(n-1) + 2(n-1)$ with $S(0) = 3$, $S(1) = 1$, by finding the generating fn.

Sol: Let $G(S; z)$ be the generating fn. of S.

Then $G(S, z) = \sum_{n=0}^{\infty} [S(n)] z^n$

Now $S(n) - S(n-1) - 2n + 2 = 0$, $n \geq 1$

So $\sum_{n=1}^{\infty} (S(n) - S(n-1) - 2n + 2) z^n = 0$

Hence

$0 = \sum_{n=1}^{\infty} S(n) z^n - \sum_{n=1}^{\infty} S(n-1) z^n - 2\sum_{n=1}^{\infty} n z^n + 2\sum_{n=1}^{\infty}$

$= [G(S; z) - S(0)] - z \left[\sum_{n=1}^{\infty} S(n-1) z^{n-1}\right]$

$- 2(z + 2z^2 + 3z^3 + \cdots) + 2\left[\sum_{n=0}^{\infty} z^n - 1\right]$

$= G(S; z) - 3 - z\, G(S; z) - 2z\left[\frac{1}{(1-z)^2}\right] + 2\left[\frac{1}{(1-z)} - 1\right]$

Rewriting we get

$$(1-z)\,G(S;z) = \frac{2z}{(1-z)^2} - \frac{2}{(1-z)} + 5$$

Hence

$$G(S;z) = \frac{2z}{(1-z)^3} - \frac{2}{(1-z)^2} + \frac{5}{(1-z)}$$

Some Common Recurrence Relation

In the earlier section we studied at length finite order linear recurrence relations with constant coefficients. In this section we study some common non linear recurrence relations.

Ex. 1 $S(n) = n(S(n-1)), n \geq 1$ and $S(0) = 1$. It is easy to see that $S(n) = n!$

Ex. 2 $G(K) = 2^K G(K-1), K > 0, G(0) = 1$

Now $G(K) = 2^K G(K-1)$

$= 2^K \cdot 2^{K-1} G(K-2)$

$= 2^K 2^{K-1} \cdots 2\, G(0)$

$= 2^{K+(K-1)+\cdots+1}$

$= 2^{k(k+1)/2}$

$= 2^{k(k+1)/2}$

Thus $O(n)$

Ex: 3 (worst case analysis of Binary Search Algorithm).

In binary search method we divide the file into two halves and search. Assume that a search one unit time after locating the middle of the file. Let $T(n)$ denote the time (worst time) required for searching a file with n records.

Then $T(n) = 1 + T([n/2])$ where $[n/2]$ is the integral part of n/2 ($[n/2]$ denotes the size of half the file) we assume that $T(0) = 0$.

The recurrence relation can be solved easily when $n = 2^k$, $k \geq 0$

$$T(2^k) = 1 + T([2^k/2]) = 1 + T(2^{k-1})$$

Repeating the calculation we get.

$$T(2^k) = k+1$$

When n is not of the form 2^k we proceed as follows.

Let r be a non-negative integer ∋:

$2^{r-1} \le n < 2^r$ Then

$N = (a_1 a_2 \ldots a_r)_2$, where R.H.S gives the binary representation of n.

Then $[n/2] = a_1 a_2 \ldots a_{r-1}$ therefore

$T(n) = T(a_1 a_2 \ldots a_r) = 1 + T(a_1 a_2 \ldots a_{r-1})$

$= 1 + (1 + T(a_1 a_2 \ldots a_{r-2}))$

$\vdots$

$= (r-1) + T(a_1) = (r-1) + T(1) = r$

Ex: Find $T(27)$ where $T(n)$ denotes the worst time for binary search of a file with n records.

Sol: Now $27 = 16 + 8 + 2 + 1$

Hence $27 = (11011)_2$

$T(27)$

$T(27) = 1 + T(1101) = 1 + (1 + T(110))$

$= 1 + 1 + (1 + T(11))$

$= 1 + 1 + 1 + T(11)$

$= 1 + 1 + 1 + 1 + T(1)$

$= 5$ since $T(1) = 1$

PRIMITIVE RECURSIVE FUNCTION

One of the basic question in Theoritical computer science is which functions can be computed by a computer. In 1934 Kurt Godel considered the primitive recursive fu: can be computed by a finite procedure (i.e. an algorithm). Around the same time Allen Turing formulated Turing machine model as a theoretical machine for performing a finite procedure (algorithm). It can be proved that partial recursive fu: (more general than primitive recursive fu:) can be Turing computable. (we are not proving this in this book)

In this section we introduce the class of primitive recursive fu: we start with three initial fu: and build up the class of primitive recursive fu:

Df 1:- A partial fu: f from x to y is a rule which assigns to every element of x atmost one element of y.

Def: A total fu: from X to Y is a rule which assign to every element of X a unique element of Y.

Ex: 1 The rule $f(x) = +\sqrt{x}$ is a partial fu: since $f(x)$ is defined only for non negative real numbers and not defined for ~~of~~ negative numbers.

Def: 2 The initial fu: over N are (i) zero fu: (ii) successor fu: (iii) Projection fu: which are defined by.

(i) zero fu: z defined by $z(x) = 0$

(ii) successor fu: s defined by $s(x) = x+1$

(iii) Projection fu: U_i^n

Def: 3 If $f_1, f_2 \ldots f_k$ are partial fu: of n variables and g is a partial fu: of k variables, then the composition of g with $f_1, f_2 \ldots f_k$ is a partial fu: ~~of a single~~ variable values. defined by

$g(f_1(x_1, x_2, \ldots x_n), f_2(x_1, x_2, \ldots x_n) \ldots$

$f_k(x_1, x_2, \ldots x_n))$

Ex: Define $f(x) = 1 + x$ and $g(x) = +\sqrt{x}$

Then $g(f(x)) = \sqrt{1+x}$

Ex: let $f_1(x,y) = x+y$, $f_2(x,y) = 2x$,

$f_3(x,y) = xy$ and $g(x,y,z) = x+y+z$

Then $g(f_1(x,y), f_2(x,y), f_3(x,y)) = g(x+y,$

$2x, xy) = x+y+2x+xy$

Thus the composition of g with f_1, f_2, f_3 is given by a fn h defined by $h(x,y) = x+y+2x+xy$.

Def: 4

A Function f defined over N is defined by recursion if there exists a constant k, $(k \in N)$, and a fn $h(x,y)$ s.t.

$f(0) = k, \quad f(n+1) = h(n, f(n))$ $\longrightarrow$ (1)

Ex:- $f(0) = 1, \; f(n+1) = h(n, f(n))$ where $h(x,y) = S(x)^* y$

As $S(x) = x+1$, $h(n, f(n)) = (n+1)^* f(n) = (n+1)\, n!$

$= (n+1)!$

Df:5 A fu: f of n+1 variables is defined by recursion if there exists a fu: g of n variables, and a fu: h of n+2 variables, and f is defined as follows

$$f(x_1, x_2 \dots x_n, 0) = g(x_1, x_2 \dots x_n) \text{ --- (I)}$$

$$f(x_1, x_2 \dots x_n, y+1) = h(x_1, x_2 \dots x_n, y, f(x_1, x_2 \dots x_n, y)) \text{ --- (II)}$$

Now we are ready to define a primitive recursive fu: over N.

Df: 6 A total fu: f over N is primitive recursive if

(a) It is one of the three initial function or

(b) it can be obtained by applying composition and recursion finite number of times to the set of initial fu:

Ex: 5 Show that $f(x,y) = x+y$, $x, y \in N$ is primitive recursive.

Sol: Now $x + (y+1) = (x+y)+1$ --- (1)

L.H.S of ① can be expressed in terms of f_1. R.H.S of ① can be expressed in terms of the Successor for S. Now let us use recursion.

Define $f_1(x,0) = x = U_1^1(x)$

$f_1(x,y+1) = S(U_3^3(x,y,f_1(x,y)))$ $\left(\because U_3^3(x,y,z) = z \ \& \ S(x) = x+1 \right)$

Now U_1^1, U_3^3, S are initial fn's. f_1 is got by composition SU_3^3 is got by recursion for the fn's by applying

U_1^1, U_3^3 and S

Hence f_1 is primitive recursive

Worked Ex:

WE:1:- Show that $f(x,y) = x*y$ is a primitive recursive fn?

So: $f(x,0) = x*0 = 0 \longrightarrow ①$

$f(x,y+1) = x*(y+1) = (x*y)+x \longrightarrow ②$

Comparing ① & ② with I and II of defn: S, we can write

$f(x,0) = Z(x) \longrightarrow ③$

~~$f(x,y) = x$~~ $f(x,y+1) = f_1(U_3^3(x,y,f(x,y)),$

$U_1^3(x,y,f(x,y))) \longrightarrow ④$

where $f_1(x,y) = x+y$, which is primitive recursive.

Taking $g = z$ and h defined by

$h(x,y,z) = f_1(U_2^3(x,y,z), U_1^3(x,y,z))$

we see that (a), (b) define f_1 by recursion. As z is an initial fn, $g = z$ is primitive recursive.

As h is defined using composition of f_1 which is primitive recursive, U_2^3, U_1^3 which are initial fn, h is primitive recursive. Hence f_2, obtain from g and h, using recursion, is primitive recursive.

(By Note following 5).

WE2:- Show that $f(x,y) = x^y$ is Primitive recursive.

Sol: Now $f(x,0) = x^0 = 1$

$f(x,y+1) = x^{y+1} = x^x x^y = x^x f(x,y)$

Define $f(x,0) = 1$

$f(x,y+1) = x^x f(x,y)$

$= U_3^3(x, y, f(x,y)) * U_3^3(x, y, f(x,y))$

Now $f(x,0) = S(Z(x))$ (Note S is a Prim recur)

$f(x, y+1) = h(x, y, f(x,y))$ where

$h(x,y,z)$ is defined by $U_3^1(x,y,z) * U_3^3(x,y,z)$

U_3^1, U_3^3 are initial fn and $f_2(x,y) = x*y$ is primitive recursive, we see that f is defined by applying recursion to primitive recursive fn $S(Z(x))$ and h. Hence f is primitive recursive fn (By Note following Ex 5)

E.X.3: Show that the following fn over N are primitive recursive

(i) Constant fn over N (ii) predecessor fn

(iii) proper subtraction fn (iv) zero test fn

(v) odd & even parity fn.

Sol:

(i) Let $f(x) = k$ be a given constant fn. Define $f(0) = k$, $f(n+1) =$ $U_2^2(n, f(n))$. As f is defined by using recursion on the initial fn: U_2^2, f is primitive recursive.

(ii) The predecessor fn: $p(x)$ defined by $p(x) = x - 1$, if $x \neq 0$ and $p(0) = 0$.

Define p by $p(0) = z(0)$, $p(x+1) = U_1^2(x, p(x))$

Hence p is Primitive recursive.

~~(iii) Proper subtraction fn: $\dot{-}$ is defined by $x \dot{-} y = x - y$ if~~

(iii) Proper subtraction fn: $\dot{-}$ is defined by $x \dot{-} y = x - y$ if $x \geq y$ &

$x \dot{-} y = 0$ if $x < y$. Define $x \dot{-} 0 = x$ &

$x \dot{-} (y+1) = p(x \dot{-} y)$ Here p is the predecessor. fn: As the fn: $\dot{-}$ is defined by using the primitive recursive recursion on the primitive fn: p (see (ii)) it is primitive recursive.

(iv) The zero-test fn: $\overline{sg}$ is defined by $\overline{sg}(0) = 1$, $\overline{sg}(x) = 0$ if $x > 0$

Define $\overline{sg}(0) = S(z(0))$, $\overline{sg}(x+1) = z(U_2^2(x, \overline{sg}(x)))$.

This fn: is Primitive recursive.

to odd and even parity fn (denoted by P_r) P_r is defined $P_r(0) = P_r(2) = \dots = 0$ and $P_r(1) = P_r(3) = \dots = 1$. Define P_r by $P_r(0) = z(0)$, $P_r(x+1) = Sg(U_2^2(x, P_r(x)))$.

As P_r is defined using recursion on the primitive recursive fn. Sg. P_r is primitive recursive.

NE1 Show that if $f(x,y)$ defines the

US remainder upon division of y by x then it is a primitive fn.

Sol: $f(x,0) = 0$. Also $f(x,y)$ increased by 1 when y is increased by 1 until the value becomes equals to x, in which case it is equated to zero and the process continuous. Now let us define f using recursion on known primitive recursive and initial fn. f is defined by

$f(x,0) = 0$, $f(x, y+1) = S(f(x,y))^{*}\,sgn(x - S(f(x,y)))$

Where sgn denotes the sgn fn. defined by $sgn(0) = z(0)$, $sgn(x+1) = S(z(U_2^2(x, sgn(x))))$

Sign funⁿ is primitive recursive (verify)
Hence f is also primitive recursive
since f is got by using recursion
on known primitive recursive fuⁿ
and composition.

Recursive And Partial Recursive Functions

Before defining recursive and
partial recursive fuⁿ, we introduce
some more definitions.

Def: 1

Let $g(x_1, x_2 \ldots x_n, y)$ be a total
fuⁿ over N. g is a regular fuⁿ
if there exist some $y_0 \in N$ ∋:
$g(x_1, x_2, \ldots, y_0) = 0$ ∀ values $x_1, x_2 \ldots x_n$ in N

Ex: 1 $g(x, y) = \min(x, y)$ is a regular fuⁿ
since $g(x, 0) = 0$ ∀ $x \in N$

Def: 2

A fuⁿ $f(x_1, x_2 \ldots x_n)$ over N is
defined from a total fuⁿ $g(x_1, x_2, \ldots x_n, y)$
by minimization if

(a) $f(x_1, x_2, \dots, x_n)$ is the least value of all y's $\ni$ $g(x_1, x_2, \dots, x_n, y)$

if it exists. The least value is denoted by $\mu_y(g(x_1, x_2, \dots, x_n, y) = 0)$

(b) $f(x_1, \dots, x_n)$ is undefined if there is no $y \ni g(x_1, x_2, \dots, x_n, y)$.

Def 1: A fn is recursive if it can be obtained from the initial fn by a finite number of applications of composition, recursion and minimization over regular fn.

Def 2: A fn is partial recursive if it can be obtained from the initial fn by a finite number of appli[cations] of composition, recursion and minimi[zation]

It is assumed that we are considering only set whose elements are natural numbers or sets of n-tuples of the natural numbers.

To each such set A we can define the characteristic fn χ_A.

The characteristic fun χ_A assigns the value 1 for all elements of A and assings the value 0 for other.

For example if $A = \{2, 4, 8, 16, 32\}$. Then $\chi_A(x) = 1$, $\forall$ $x = 2, 4, 8, 16$, or 32 and $\chi_A(x) = 0$ $\forall$ natural numbers $x \neq 2, 4, 8, 16, 32$.

Def 5

A set A is called recursive (partial recursive) if the characteristic fun χ_A is recursive (partial recursive)

Ex

Show that $f(x) = x/2$ is partial recursive.

Sol: Let $g(x,y) = |2y - x|$, $2y - x = 0$ for some y only when x is even. Let $f_1(x) = \mu_y(|2y - x| = 0)$ Then f_1 is defined only for even values of x and is equal to $x/2$. When x is odd, $f_1(x)$ is not defined so f_1 and hence f is partial recursive.

W.E. [illegible] Let $[\sqrt{x}]$ be the integral part of $\sqrt{x}$. Show that $[\sqrt{x}]$ is recursive.

Sol: We can note that $(y+1)^2 - x$ is zero for $(y+1)^2 \leq x$ and non-zero for $(y+1)^2 > x$. Hence $\bar{sg}((y+1)^2 - x) = 1$ if $(y+1)^2 \leq x$. Now $[\sqrt{x}]$ is the smallest value of y for which $(y+1)^2 > x$.

Hence $[\sqrt{x}] = \mu_y (\bar{sg}((y+1)^2 - x) = 0)$

$\bar{sg}((y+1)^2 - x)$ is a regular fn of x. As $[\sqrt{x}]$ is got by minimization of a regular fn, $[\sqrt{x}]$ is recursive.

W.E. 3 Show that the set of divisors of a positive integer n is recursive.

Sol: A set is recursive if its characteristic fn is recursive. Now a number $x \leq n$ is a divisor of n if and only if $|x * i - n| = 0$ for some fixed i, $1 \leq i \leq n$. Also $|x * i - n|$ is non-zero for all i, $1 \leq i \leq n$, if x is not a divisor of n. Let χ_B denote the characteristic fn of the set of all divisors of n.

Then

$$\chi_B(x) = \sum_{i=1}^{n} \bar{s}g\,(x*i - n)$$

(Note that i is a divisor of n $\Leftrightarrow$ $|x*i - n| = 0 \Leftrightarrow \bar{s}g\,|x*i - n| = 1$. As χ_B is got as a finite sum of primitive primitive recursive fn it a recursive.

Ex 2.9 If A denotes Ackermann's fn evaluate a) $A(1,1)$ b) $A(1,2)$ c) $A(0,1)$ d) $A(2,2)$ e) $A(2,1)$ f) $A(3,2)$

Sol Recall the definition of A from

Ex: 2.81

$A(0,y) = y+1$ ⟶ (1)

$A(x+1, 0) = A(x,1)$ ⟶ (2)

$A(x+1, y+1) = A(x, A(x+1, y))$ ⟶ (3)

a) $A(1,1) = A(0+1, 0+1) = A(0, A(1,0))$. by (3)

$= A(0, A(0,1))$. by (2)

$= A(0,2)$ as by (1) $A(0,1) = 1+1 = 2$

$= 3$ by (1)

Hence $A(1,1) = 3$.

b) $A(1,2) = A(0+1, 1+1) = A(0, A(1,1))$ by (3)

$= 4$ by (1)

Hence $A(1,2) = 4$.

c) $A(2,1) = A(1+1, 0+1) = A(1, A(2,0))$

$= A(1, A(1,1)) = A(1,3) = A(0+1,$ [illegible]

$= A(0, A(1,2)) = A(0,4) = 5$

Hence $A(2,1) = 5$

d) $A(2,2) = A(1+1, 1+1) = A(1, A(2,1)) =$ [illegible]

$= A(0+1, 4+1) = A(0, A(1,4))$

$= 1 + A(1,4) = 1 + A(0+1, 3+1) = 1 +$ [illegible]

$= 1 + 1 + A(1,3) = 1 + 1 + A(1+0, 2+0)$

$= 2 + A(0, A(1,2)))$

$= 2 + 1 + A(1,2)$ (from (3))

$= 3 + 4$ (from (b))

$= 7$

Hence $A(2,2) = 7$

e) For proving this we need the result

$A(1,y) = y+2 \longrightarrow (I)$

Now $A(1,y) = A(0+1, y-1+1) = A(0, A(1, y-1))$

$= 1 + A(1, y-1) = \cdots = y-1+A(1,1)$

$= y-1+3 = y+2$ By (2))

Now $A(3,1) = A(2+1, 0+1) = A(2, A(3,0))$

$= A(2, A(2,1))$ by(5)

$= A(2,5)$

$= A(1+1, 4+1) = A(1, A(2,4))$

$= 2 + A(2,4)$ from ④

$= 2 + A(1, A(2,3)) = 2+2+A(2,3)$

$= 4 + A(1, A(2,2)) = 4+2+A(2,2)$

by ④

$= 6+7 = 13$.

Hence $A(3,1) = 13$

f) $A(3,2) = A(2+1, 1+1) = A(2, A(3,1))$

$= A(2,13)$ from ⓒ

$= A(1, A(2,12)) = 2 + A(1,13)$

(by (i))

$= 2+2+13$ (by (i))

$= 17$

Hence $A(3,2) = 17$